Biometrical Letters
Vol. 46 (2009), No. 1, 1-14

Binary operations on prime basis factorials

Vera Jesus[1], João Tiago Mexia[1], Sandra Oliveira[2]

[1]Faculty of Science and Technology-Mathematics Department,
New University of Lisbon, Monte da Caparica 2829-516 Caparica,
Portugal, e-mail: veramjesus@gmail.com, jtm@fct.unl.pt
[2]College of Business Administration-Polytechnic Institute of Setubal,
Estefanilha 2914-503 Setúbal, Portugal, e-mail: soliveira@esce.ips.pt

Summary

Binary operations and commutative Jordan algebras may be used to define product models in which the treatments are a combination of those in the initial models and nested models in which each treatment of a model nests all the treatments of another model. This technique is applied here to prime basis factorials. The notion of a model strictly associated with a commutative Jordan algebra is introduced and applied to prime basis factorials.

Key words: commutative Jordan algebras, binary operations, prime basis factorials, associated models

1. Introduction

Among orthogonal models, the prime basis factorials and their fractional replicates are distinguished for their flexibility.

In our study of these models, we introduce the notion of a model strictly associated with a Commutative Jordan Algebra, CJA, which enables us to obtain in a straightforward way UMVUE (uniformely minimum variance unbiased estimators) and, moreover, enables the use of binary operations to build more complex designs, using prime basis factorials and their fractional factorials as building blocks. For the definitions of prime basis factorials, fractional factorials and blocks, see for instance Dey & Mukerjee (1999), Montgomery (1997) or Mukerjee & Wu (2006).

Firstly we will study the algebraic structure of such models, showing how to associate them with commutative Jordan algebras and with orthogonal matrices.

Next we use binary operations on commutative Jordan algebras - see Fonseca et al. (2006) - to study the crossing and nesting of prime basis factorials.

2. Commutative Jordan algebras

Commutative Jordan algebras are linear spaces constituted by symmetric matrices that commute and contain the squares of every matrix in the space. Jordan algebras were introduced by Jordan et al. (1934) to provide an algebraic foundation for Quantum Mechanics. Later, Seely used these algebras to carry out inference, moreover Seely (1971) proved that each CJA has a unique basis $\{\mathbf{Q}_1, ..., \mathbf{Q}_w\}$, constituted by mutually orthogonal projection matrices. This basis is called the principal basis of the algebra. If

$$\mathbf{Q}_1 = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^{'} = \frac{1}{n}\mathbf{J}_n \tag{1}$$

where $\mathbf{1}_n^{'}$ is a matrix $n \times 1$ with elements equal to 1, the CJA will be regular and if

$$\sum_{j=1}^{w} \mathbf{Q}_j = \mathbf{I}_n \tag{2}$$

it will be complete. In what follows we will only consider complete and regular CJA. If the vectors of matrices $\mathbf{A}_j, j = 1, ..., w$ constitute an orthonormal basis for the range of $\mathbf{Q}_j, j = 1, ..., w$ we will have,

$$\mathbf{Q}_j = \mathbf{A}_j^{'}\mathbf{A}_j, \quad j = 1, ..., w. \tag{3}$$

Namely,

$$\mathbf{A}_1 = \frac{1}{\sqrt{n}}\mathbf{1}_n^{'} \tag{4}$$

when CJA is regular. Moreover, the matrix

$$\mathbf{P} = [\mathbf{A}_1^{'}...\mathbf{A}_w^{'}]^{'} \tag{5}$$

is orthogonal. We say that it is an orthogonal matrix associated with the CJA with principal basis $\mathcal{N}(\mathcal{A}) = \{\mathbf{Q}_1, ..., \mathbf{Q}_w\}$.

3. Binary operations

We represent by $\otimes$ the Kronecker matrix product. This matrix operation is studied for instance in Steeb (1991).

Given $\{\mathbf{Q}_{l,1}, ..., \mathbf{Q}_{l,w_l}\}$ the principal basis constituted by $n_l \times n_l$ matrices of the complete and regular CJA $\mathcal{A}_l$, $l = 1, 2$,

$$\mathbf{Q}_{\mathbf{j}} = \mathbf{Q}_{1,j_1} \otimes \mathbf{Q}_{2,j_2}, \quad j_1 = 1, ..., w_l, \ l = 1, 2, \tag{6}$$

will be the principal basis of the CJA $\mathcal{A}_1 \otimes \mathcal{A}_2$.

The matrices of $\mathcal{A}_1 \otimes \mathcal{A}_2$ are the Kronecker product of the matrices of $\mathcal{A}_1$ by those of $\mathcal{A}_2$.

If $\mathbf{P}_l$ is an orthogonal matrix associated with $\mathcal{A}_l$, $l = 1, 2$, $\mathbf{P}_1 \otimes \mathbf{P}_2$ will be an orthogonal matrix associated with $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Moreover the restricted $\mathcal{A}_1 \circledast \mathcal{A}_2$ Kronecker product of the CJA $\mathcal{A}_1$ and $\mathcal{A}_2$ will be the CJA with principal basis constituted by the

$$\mathbf{Q}_{1,j} \otimes \frac{1}{n_2}\mathbf{J}_{n_2}, \quad j = 1, ..., w_1, \tag{7}$$

and the

$$\mathbf{I}_{n_1} \otimes \mathbf{Q}_{2,j}, \quad j = 2, ..., w_2, \tag{8}$$

where it is assumed that the matrices in $\mathcal{A}_l$ are $n_l \times n_l$, $l = 1, 2$.

If $\mathcal{A}_l$, $l = 1, 2$, are regular and complete with

$$\begin{cases} \mathbf{Q}_{l,1} = \frac{1}{n_l}\mathbf{J}_{n_l}, \quad l = 1, 2 \\ \sum_{j=1}^{w_l} \mathbf{Q}_{l,j} = \mathbf{I}_{n_l}, \quad l = 1, 2 \end{cases}, \tag{9}$$

$\mathcal{N}(\mathcal{A}_1 \circledast \mathcal{A}_2) = \mathcal{N}(\mathcal{A}_1 \otimes \mathbf{J}_{n_2}) \cup [\mathbf{I}_{n_1} \otimes \mathcal{N}(\mathcal{A}_2^*)]$ with $\mathcal{N}(\mathcal{A}_2^*) = \{\mathbf{Q}_{2,2}, ..., \mathbf{Q}_{2,w_2}\}$, will be the principal basis of the CJA $\mathcal{A}_1 \circledast \mathcal{A}_2$. This new CJA will also be regular and complete.

While the operation $\otimes$ will be used for model crossing, so the treatments of the final model will be the combinations of the treatments of the initial models, the operation $\circledast$ will be used for model nesting, all the treatments of the second model being nested inside each treatment of the first model.

We can see, in Fonseca et al. (2006), that both operations $\otimes$ and $\circledast$ are associative.

Another interesting application of $\circledast$ will be when we consider r observations per treatment. Let $\mathcal{A}(r)$ be the CJA with principal basis $\left\{\frac{1}{r}\mathbf{J}_r, \overline{\mathbf{J}}_r\right\}$ where $\overline{\mathbf{J}}_r = \mathbf{I}_r - \frac{1}{r}\mathbf{J}_r$. Then, if $\mathcal{A}$ is the relevant algebra when $r = 1$, when $r > 1$ the relevant algebra will be $\mathcal{A} \circledast \mathcal{A}(r)$.

Moreover, if

$$\mathbf{P}_j = \begin{bmatrix} \frac{1}{\sqrt{n_l}}\mathbf{1}'_{n_l} \\ \mathbf{K}_{n_l} \end{bmatrix}, \quad l = 1, 2, \tag{10}$$

is the orthogonal matrix associated with $\mathcal{A}_l$, $l = 1, 2$,

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 \otimes \frac{1}{n_2}\mathbf{J}_{n_2} \\ \mathbf{I}_{n_1} \otimes \mathbf{K}_2 \end{bmatrix} \tag{11}$$

will be the orthogonal matrix associated with $\mathcal{A}_1 \circledast \mathcal{A}_2$.

4. Strictly associated models

Let $\mathbf{P} = [\mathbf{A}_1^{'}...\mathbf{A}_w^{'}]'$ be an orthogonal matrix associated with a commutative Jordan algebra $\mathcal{A}$ constituted by $n \times n$ matrices with principal basis $\{\mathbf{Q}_1, ..., \mathbf{Q}_w\}$.

We put $\mathbf{z} \sim N(\boldsymbol{\eta}, \mathbf{W})$ when $\mathbf{z}$ is normal with mean vector $\boldsymbol{\eta}$ and covariance matrix $\mathbf{W}$.

If we have r repetitions, the model

$$\mathbf{Y} = \sum_{j=1}^{m}\left(\mathbf{A}'_j \otimes \frac{1}{\sqrt{r}}\mathbf{1}_r\right)\boldsymbol{\eta}_j + \sum_{j=m+1}^{w}\left(\mathbf{A}'_j \otimes \frac{1}{\sqrt{r}}\mathbf{1}_r\right)\boldsymbol{\eta}_j^* + \mathbf{e} \tag{12}$$

will be strictly associated with the CJA $\mathcal{A} \circledast \mathcal{A}(r)$. We assume that the vectors $\boldsymbol{\eta}_j$, $j = 1, ..., m$ are fixed and that $\boldsymbol{\eta}_j^* \sim N\left(\mathbf{0}, \sigma_j^2\mathbf{I}_{g_j}\right)$, $j = m+1, ..., w$ and $\mathbf{e} \sim N\left(\mathbf{0}, \sigma^2\mathbf{I}_n\right)$, these vectors being independent, and $g_j = rank(\mathbf{A}_j) = rank(\mathbf{Q}_j)$.

$\mathbf{Y}$ will be normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{V}$, given by

$$\begin{cases} \boldsymbol{\mu} = \sum_{j=1}^{m}\left(\mathbf{A}'_j \otimes \frac{1}{\sqrt{r}}\mathbf{1}_r\right)\boldsymbol{\eta}_j \\ \mathbf{V} = \sum_{j=1}^{w}\gamma_j\left(\mathbf{Q}_j \otimes \frac{1}{\sqrt{r}}\mathbf{1}_r\right) + \sigma^2\mathbf{Q}^{\perp} \end{cases}, \tag{13}$$

where $\gamma_j = \sigma^2, j = 1, ..., m$, $\gamma_{m+l} = \sigma^2_{m+l} + \sigma^2, l = 1, ..., w - m$ and

$$\mathbf{Q}^{\perp} = \mathbf{I}_n \otimes \overline{\mathbf{J}}_r. \tag{14}$$

Now, see Fonseca et al. (2006),

$$\begin{cases} \mathbf{V}^{-1} = \sum\limits_{j=1}^{w} \gamma_j^{-1}\mathbf{Q}_j + (\sigma^2)^{-1}\mathbf{Q}^{\perp} \\ \det(\mathbf{V}) = \prod\limits_{j=1}^{w} \gamma_j^{g_j}(\sigma^2)^g \end{cases} \tag{15}$$

with $g = n\,(r-1)$, and taking

$$\begin{cases} \boldsymbol{\eta}_j = \left(\mathbf{A}_j \otimes \frac{1}{\sqrt{r}}\mathbf{1}'_r\right)\boldsymbol{\mu}, \quad j = 1, ..., w \\ \widetilde{\boldsymbol{\eta}}_j = \left(\mathbf{A}_j \otimes \frac{1}{\sqrt{r}}\mathbf{1}'_r\right)\mathbf{Y}, \quad j = 1, ..., w \end{cases}, \tag{16}$$

we will have $\boldsymbol{\eta}_j = \mathbf{0}$, $j = m+1, ..., w$, as well as

$$(\mathbf{Y} - \boldsymbol{\mu})'\,\mathbf{V}^{-1}\,(\mathbf{Y} - \boldsymbol{\mu}) = \sum_{j=1}^{m} \frac{\|\widetilde{\boldsymbol{\eta}}_j - \boldsymbol{\eta}_j\|^2}{\gamma_j} + \sum_{j=m+1}^{w} \frac{S_j}{\gamma_j} + \frac{S}{\sigma^2}, \tag{17}$$

where $S_j = \|\widetilde{\boldsymbol{\eta}}_j\|^2 = \mathbf{Y}'\mathbf{Q}_j\mathbf{Y}$, $j = m+1, ..., w$, and $S = \mathbf{Y}'\mathbf{Q}^{\perp}\mathbf{Y}$. Thus the density of $\mathbf{Y}$ can be written as

$$n\,(\mathbf{Y}|\boldsymbol{\mu}, \mathbf{V}) = \frac{e^{-\frac{1}{2}\left[\sum\limits_{j=1}^{m} \frac{\|\widetilde{\boldsymbol{\eta}}_j - \boldsymbol{\eta}_j\|^2}{\gamma_j} \sum_{j=m+1}^{w} \frac{S_j}{\gamma_j} + \frac{S}{\sigma^2}\right]}}{(2\pi)^{n/2}\prod_{j=1}^{w} \gamma_j^{g_j/2}(\sigma^2)^{g/2}}. \tag{18}$$

Moreover, writing $\mathbf{z} \sim \theta\chi^2_p$ to indicate that $\mathbf{z}$ is the product by θ of a central chi-square with p degrees of freedom, the

$$\begin{cases} \widetilde{\boldsymbol{\eta}}_j \sim N\left(\boldsymbol{\eta}_j, \sigma^2\mathbf{I}_{g_j}\right), \quad j = 1, ..., m \\ S_j \sim \gamma_j\chi^2_{g_j}, \quad j = m+1, ..., w \\ S \sim \sigma^2\chi^2_g \end{cases}$$

will be complete and sufficient statistics. According to the Blackwell-Lehmman-Scheffé theorem the $\widetilde{\boldsymbol{\eta}}_j, j = 1, ..., m$, the $\widetilde{\gamma}^2_j = \frac{S_j}{g_j}, j = m+1, ..., w$, and $\widetilde{\sigma}^2 = \frac{S}{g}$ as well as the $\widetilde{\sigma}^2_j = \widetilde{\gamma}^2_j - \widetilde{\sigma}^2$, $j = m+1, ..., w$ will be UMVUE.

The interest of the concept of strictly associated models is twofold. As we have seen it leads to optimal estimators and, as we now show, we can apply binary operations to strictly associated models to obtain new strictly associated models.

5. Crossing models

If we cross the models

$$\mathbf{Y}_l = \sum_{j=1}^{w_l} \left(\mathbf{A}'_{l,j} \otimes \frac{1}{\sqrt{r_l}} \mathbf{1}_{r_l} \right) \boldsymbol{\eta}_{l,j} + \mathbf{e}, \quad l = 1, 2, \tag{19}$$

strictly associated with the $\mathcal{A}_l \circledast \mathcal{A}(r), l = 1, 2$ we obtain a model

$$\mathbf{y} = \sum_{j_1=1}^{w_1} \sum_{j_2=1}^{w_2} \left(\mathbf{A}'_{1,j_1} \otimes \mathbf{A}'_{2,j_2} \otimes \frac{1}{\sqrt{r}} \mathbf{1}_r \right) \boldsymbol{\eta}_{j_1,j_2} + \mathbf{e} \tag{20}$$

strictly associated with the $(\mathcal{A}_1 \otimes \mathcal{A}_2) \circledast \mathcal{A}(r)$.

If the fixed effects parts of the initial models are

$$\sum_{j=1}^{m_l} \left(\mathbf{A}'_{l,j} \otimes \frac{1}{\sqrt{r_l}} \mathbf{1}_{r_l} \right) \boldsymbol{\eta}_{l,j}, \quad l = 1, 2, \tag{21}$$

the fixed part of the new model will be

$$\sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left(\mathbf{A}'_{1,j_1} \otimes \mathbf{A}'_{2,j_2} \otimes \frac{1}{\sqrt{r}} \mathbf{1}_r \right) \boldsymbol{\eta}_{j_1,j_2}. \tag{22}$$

6. Nested models

If we nest all treatments of the second initial model inside each treatment of the first initial model, we obtain the new model,

$$\begin{aligned} \mathbf{Y} = \; & \sum_{j_1=1}^{w_1} \left(\mathbf{A}'_{1,j_1} \otimes \tfrac{1}{\sqrt{n_2}} \mathbf{1}_{n_2} \otimes \tfrac{1}{\sqrt{r}} \mathbf{1}_r \right) \boldsymbol{\eta}_{j_1} + \\ & \sum_{j_2=1}^{w_2} \left(\mathbf{I}_{n_1} \otimes \mathbf{A}'_{2,j_2} \otimes \tfrac{1}{\sqrt{r}} \mathbf{1}_r \right) \boldsymbol{\eta}_{w_1+j_2-1} + \mathbf{e} \end{aligned} \tag{23}$$

strictly associated with the CJA $\mathcal{A}_1 \circledast \mathcal{A}_2 \circledast \mathcal{A}(r)$.

As the random effects factors do not nest fixed effects factors, if the first model has random effects factors, the second cannot have fixed effects factors. Similarly, if the second model has fixed effects factors, the first can only have fixed effects factors. Thus, representing for Fx, $[Al,\ Mt]$ the fixed effects models, [random, mixed], we will have the following possible cases:

1^{st} model	2^{nd} model
Random	Random
Mixed	Random
Fixed	Fixed , Random, Mixed

7. Prime basis factorials

We will consider only the factorial p^N design with N ($N > 1$, integer) factors each having a prime number p of levels. Numbering the levels from 0 to $p-1$, the p^N treatments may be represented by vectors $\mathbf{x} = [x_1, ..., x_N]^{'}$, where $x_j = 0, ..., p-1, j = 1, ..., N$. These vectors can be ordered by the indexes

$$l(\mathbf{x}) = 1 + \sum_{j=1}^{N} x_j p^{j-1}. \tag{24}$$

Let $\mathcal{L}_{[p]}^{N}$ be the family of the linear applications

$$L_{\mathbf{a}}(\mathbf{x}) = \mathbf{a}^{'}\mathbf{x} \tag{25}$$

whose values are obtained using modulo p arithmetic. The components of vector $\mathbf{a}$ are called the coefficients of application L. These linear applications and their coefficients constitute linear spaces. These spaces are isomorphic, then $L_{\mathbf{a}_1}(\mathbf{x}), ..., L_{\mathbf{a}_m}(\mathbf{x})$ are linearly independent if and only if $\mathbf{a}_1, ..., \mathbf{a}_m$ are linearly independent. Since these spaces have dimension N there cannot be more than N linearly independent applications. We will single out the reduced applications where the first non-null coefficient is equal to 1. It is easy to see that any two reduced applications are linearly independent. Let $\mathcal{L}_{r[p]}^{N}$ be the family of such applications. There are

$$w = \frac{p^N - 1}{p - 1} \tag{26}$$

reduced applications.

With $L_1, ..., L_m$ linearly independent reduced applications,

$$[\mathbf{L}|\mathbf{b}] = [L_1, ..., L_m|b_1, ..., b_m] = \{\mathbf{x} : L_i(\mathbf{x}) = b_i, \quad i = 1, ..., m\} \tag{27}$$

will be a set of treatments called a block. Since the system of equations $L_i(\mathbf{x}) = b_i, i = 1, ..., m$, enables us to express m components of $\mathbf{x}$ as linear combination of the remaining components, in every $[\mathbf{L}|\mathbf{b}]$ there will be p^{N-m} treatments and there will be p^m blocks.

We may order the reduced applications $L \in \mathcal{L}^N_{r[p]}$ according to the increase of indexes $l(\mathbf{a})$. Thus if $l(\mathbf{a}_1) < l(\mathbf{a}_2)$ we will have $k(\mathbf{a}_1) < k(\mathbf{a}_2)$, $k(\mathbf{a}) = 1, ..., w$. To each $L \in \mathcal{L}^N_{r[p]}$ we can associate a $p \times p^N$ matrix $C(L)$ with elements

$$c_{i,j}(L) = \begin{cases} 0, & L(\mathbf{x}_j) \neq i-1 \\ 1, & L(\mathbf{x}_j) = i-1 \end{cases}, \tag{28}$$

$i = 1, ..., p$ and $j = 1, ..., p^N$. It is easy to see that

$$\mathbf{C}(L)\mathbf{C}(L)' = p^{N-1}\mathbf{I}_p \tag{29}$$

since L takes each of its values for p^{N-1} treatments. Moreover if L_1 and L_2 are linearly independent

$$\mathbf{C}(L_1)\mathbf{C}(L_2)' = p^{N-2}\mathbf{J}_p \tag{30}$$

since $[L_1, L_2|b_1, b_2]$ contains p^{N-2} treatments whatever the pair (b_1, b_2). Let $\mathbf{K}$ be a $(p-1) \times p$ matrix obtained by deleting the first row equal to $\frac{1}{\sqrt{p}}\mathbf{1}'_p$ of an orthogonal matrix. Then with $q = p^{\frac{N-1}{2}}$ we consider the matrices

$$\mathbf{B}(L) = \frac{1}{q}\mathbf{K}\mathbf{C}(L), \quad L \in \mathcal{L}^N_{r[p]}. \tag{31}$$

We now prove the following:

Proposition 1. *The matrix*

$$\mathbf{P}(p^N) = \left[\frac{1}{p^{\frac{N}{2}}}\mathbf{1}_{p^N}\mathbf{B}(L_1)'...\mathbf{B}(L_w)'\right]' \tag{32}$$

is orthogonal and it is associated with the CJA $\mathcal{A}(p^N)$ with principal basis

$$\mathcal{N}(p^N) = \left\{\frac{1}{p^N}\mathbf{J}_{p^N}, \mathbf{Q}(L_1), ..., \mathbf{Q}(L_w)\right\} \tag{33}$$

where

$$\mathbf{Q}(L_j) = \mathbf{B}(L_j)'\mathbf{B}(L_j), \quad j = 1, ..., w. \tag{34}$$

Proof. Firstly, according to (13) we have

$$\mathbf{B}(L_j)\mathbf{B}(L_j)^{'} = \frac{1}{q^2}\mathbf{K}\mathbf{C}(L_j)\mathbf{C}(L_j)^{'}\mathbf{K}^{'} = \frac{1}{p^{N-1}}\mathbf{K}\left(p^{N-1}\mathbf{I}_p\right)\mathbf{K}^{'} = \mathbf{K}\mathbf{K}^{'} = \mathbf{I}_p,$$

and for $i \neq j$,

$$\begin{aligned}\mathbf{B}(L_i)\mathbf{B}(L_j)^{'} &= \frac{1}{q^2}\mathbf{K}\mathbf{C}(L_i)\mathbf{C}(L_j)^{'}\mathbf{K}^{'} = \frac{1}{p^{N-1}}\mathbf{K}\left(p^{N-2}\mathbf{1}_p\mathbf{1}_p^{'}\right)\mathbf{K}^{'}\\ &= \frac{1}{p}\left(\mathbf{K}\mathbf{1}_p\right)\left(\mathbf{K}\mathbf{1}_p\right)^{'} = \mathbf{0}_{p-1,p-1}.\end{aligned}$$

Thus the $\frac{1}{p^N}\mathbf{J}_{p^N}, \mathbf{Q}(L_1), ..., \mathbf{Q}(L_w)$ are symmetric, idempotent and mutually orthogonal, so they will constitute the principal basis of CJA $\mathcal{A}(p^N)$. □

The model strictly associated with the CJA $\mathcal{A}(p^N)$ is

$$\mathbf{Y} = \mathbf{1}_{p^N}\mu + \sum_{L\in\mathcal{L}^N_{r[p]}} \left(\mathbf{B}(L)^{'}\beta(L)\right) + \mathbf{e}. \tag{35}$$

Since the matrices

$$\mathbf{M}(L) = \mathbf{B}(L^{'})\mathbf{B}(L), \quad L \in \mathcal{L}^N_{r[p]} \tag{36}$$

commute, the model is orthogonal; see Fonseca et al. (2006).

If we take r observations per treatment we will have a model strictly associated with the CJA $\mathcal{A}(p^N) \circledast \mathcal{A}\,(r)$.

If we take only the treatments in one of the blocks $[L_1, ..., L_m|b_1, ..., b_m]$, we will have a fractional replicate $\frac{1}{p^m} \times p^N$. Usually we take $L_1, ..., L_m \in \mathcal{L}^N_{r[p]}$. Now $\{L_1, ..., L_m\}$ may be completed to give a basis $\{L_1, ..., L_m, L_{m+1}, ..., L_N\}$ to $\mathcal{L}^N_{[p]}$. In the subspace generated by the $L_{m+1}, ..., L_N$ there will be

$$w^+ = \frac{p^{N-m} - 1}{p - 1} \tag{37}$$

reduced linear applications. Let $\mathcal{L}^{+N}_{r[p]}$ be the set of these applications. With $L^* \in \mathcal{L}^{+N}_{r[p]}$, $C(L^* \mid \mathbf{L})$ will be the sub-matrix of $C(L^*)$ constituted by the column vectors associated with the $\mathbf{x} \in [\mathbf{L} \mid \mathbf{b}]$. We can use the indexes of the coefficient vectors for the applications in $L^* \in \mathcal{L}_{r[p]}^{+N}$ to order them from $1, ..., w^+$. Considering the matrices

$$\mathbf{B}(L_i^+) = \frac{1}{q}\mathbf{K}\mathbf{C}(L_i^+), \quad i = 1, ..., w^+ \tag{38}$$

with $L_i^+ \in \mathcal{L}_{r[p]}^{+N}$, $q = p^{\frac{N^+-2}{2}}$, $N^+ = N - m$ and $\mathbf{C}(L_i^+) = \mathbf{C}(L_i^+ \mid \mathbf{L})$. We now prove:

Proposition 2. *The matrix*

$$\mathbf{P}(p^{N+}) = \left[\frac{1}{p^{\frac{N^+}{2}}}\mathbf{1}_{p^{N+}}\mathbf{B}(L_1^+)^{'}...\mathbf{B}(L_w^+)^{'}\right]^{'} \tag{39}$$

is orthogonal and is associated with the CJA $\mathcal{A}(p^{N+})$ with principal basis

$$\mathcal{N}(p^{N^+}) = \left\{\frac{1}{p^{N^+}}\mathbf{J}_{p^N+}, \mathbf{Q}(L_1^+), ..., \mathbf{Q}(L_w^+)\right\} \tag{40}$$

where $\mathbf{Q}(L_j^+) = \mathbf{B}(L_j^+)^{'}\mathbf{B}(L_1^+), \quad j = 1, ..., w^+$.

Proof. Firstly, we have

$$\mathbf{B}(L_j^+)\mathbf{B}(L_j^+)^{'} = \frac{1}{q^2}\mathbf{K}\mathbf{C}(L_j^+)\mathbf{C}(L_j^+)^{'}\mathbf{K}^{'} = \frac{1}{p^{N-1}}\mathbf{K}\left(p^{N-1}\mathbf{I}_p\right)\mathbf{K}^{'} = \mathbf{K}\mathbf{K}^{'} = \mathbf{I}_p,$$

and for $i \neq j$ we have

$$\begin{aligned}\mathbf{B}(L_i^+)\mathbf{B}(L_j^+)^{'} &= \frac{1}{q^2}\mathbf{K}\mathbf{C}(L_i^+)\mathbf{C}(L_j^+)^{'}\mathbf{K}^{'} = \frac{1}{p^{N-1}}\mathbf{K}\left(p^{N-2}\mathbf{1}_p\mathbf{1}_p^{'}\right)\mathbf{K}^{'} \\ &= \frac{1}{p}\left(\mathbf{K}\mathbf{1}_p\right)\left(\mathbf{K}\mathbf{1}_p\right)^{'} = \mathbf{0}_{p-1,p-1}.\end{aligned}$$

Thus the $\frac{1}{p^N}\mathbf{J}_{p^N}, \mathbf{Q}(L_1^+), ..., \mathbf{Q}(L_w^+)$ are symmetric, idempotent and mutually orthogonal, so they will constitute the principal basis of CJA $\mathcal{A}(p^N)$. □

As before, if we take r observations per treatment we will have a model strictly associated with the CJA $\mathcal{A}(p^{N^+}) \circledast \mathcal{A}(r)$.

We now can apply the binary operations to prime basis factorials and their fractional replicates as building blocks of more complex models. In this way we overcome the usual limitation - see Mukerjee & Wu (2006) - of the factors having a prime or prime-power number of levels.

For instance $\left(\frac{1}{p_1^m} \times p_1^N\right) \otimes \left(\frac{1}{p_2^m} \times p_2^N\right)$ may be treated as a model whose factors have $p_1 p_2$ levels.

Once applied the binary operations we obtain models strictly associated with CJA. We can then use the results in section 3 to carry out the inference.

8. Application

Let us now apply our results to a computer experiment. In this experiment we had two six-level factors. Each of these factors was obtained by aggregating a three-level factor with a two-level factor. The levels of the initial factors were represented by x_1 and x_2 for the two-level factors and by z_1 and z_2 for the three-level factors. We aggregated the first [second] factors in each pair.

We used the reduced linear applications $x_1 + x_2$ and $z_1 + 2z_2$ to generate blocks. In the following table we present those blocks, the values for the different treatments as well as the corresponding indices.

Table 1. Designs, results and indices.

b_1	b_2	x_1	x_2	z_1	z_2	*results*	*indices*	b_1	b_2	x_1	x_2	z_1	z_2	*results*	*indices*
0	0	0	0	0	0	5.1	1	1	0	0	1	0	0	5.5	3
		0	0	1	1	7.3	17			0	1	1	1	7.6	19
		0	0	2	2	8.7	33			0	1	2	2	9.4	35
		1	1	0	0	7.2	4			1	0	0	0	5.4	2
		1	1	1	1	9.2	20			1	0	1	1	7.7	18
		1	1	2	2	11.3	36			1	0	2	2	9.6	34
0	1	0	0	0	2	6.2	9	1	1	0	1	0	2	8.3	11
		0	0	1	0	5.1	13			0	1	1	0	7.4	15
		0	0	2	1	7.3	29			0	1	2	1	9.5	31
		1	1	0	2	8.6	12			1	0	0	2	8.2	10
		1	1	1	0	9.7	16			1	0	1	0	7.1	14
		1	1	2	1	11.8	32			1	0	2	1	9.3	30
0	2	0	0	0	1	5.1	5	1	2	0	1	0	1	7.4	7
		0	0	1	2	7.3	21			0	1	1	2	10.3	23
		0	0	2	0	6.1	25			0	1	2	0	9.4	27
		1	1	0	1	7.4	8			1	0	0	1	7.2	6
		1	1	1	2	9.1	24			1	0	1	2	10.3	22
		1	1	2	0	8.2	28			1	0	2	0	9.2	26

For the reduced linear applications connected with the two-level factors we had the row matrices:

$$\mathbf{A}(x_1) = \frac{1}{2}[\ -1 \quad -1 \quad 1 \quad 1\ ],$$

$$\mathbf{A}(x_2) = \frac{1}{2}[\ -1 \quad 1 \quad -1 \quad 1\ ],$$

$$\mathbf{A}(x_3) = \frac{1}{2}[\ -1 \quad 1 \quad 1 \quad -1\ ].$$

For the reduced linear applications connected with the three-level factors we had the associated matrices:

$$\mathbf{A}(z_1) = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

$$\mathbf{A}(z_2) = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix},$$

$$\mathbf{A}(z_1+z_2) = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 0 \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} \end{bmatrix},$$

$$\mathbf{A}(z_1+2z_2) = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & 1 & 1 & 0 & -1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Besides these we also considered the row matrices $\frac{1}{3}\mathbf{1}_9'$ and $\frac{1}{2}\mathbf{1}_4'$ for the two groups of factors.

To complete the sums of squares we nested the two-level factors inside the three-level factors and ordered the results accordingly. The correspondent indices are those presented in Table 1.

If we had not merged the factors, the origins of variation would be connected to one or both groups of factors. For the three-level factors we would have the sums of squares

$$S(z_1) = \left\| \left(\mathbf{A}(z_1) \otimes \frac{1}{2}\mathbf{1}_4' \right) \mathbf{Y} \right\|^2 = 33.455,$$

$$S(z_2) = \left\| \left(\mathbf{A}(z_2) \otimes \frac{1}{2}\mathbf{1}_4' \right) \mathbf{Y} \right\|^2 = 19.995,$$

$$S(z_1+z_2)=\left\|\left(\mathbf{A}(z_1+z_2)\otimes\frac{1}{2}\mathbf{1}_4'\right)\mathbf{Y}\right\|^2=0.335,$$

$$S(z_1+2z_2)=\left\|\left(\mathbf{A}(z_1+2z_2)\otimes\frac{1}{2}\mathbf{1}_4'\right)\mathbf{Y}\right\|^2=2.66,$$

for the two-level factors we had

$$S(x_1)=\left\|\left(\frac{1}{3}\mathbf{1}_9'\otimes\mathbf{A}(x_1)\right)\mathbf{Y}\right\|^2=17.500,$$

$$S(x_2)=\left\|\left(\frac{1}{3}\mathbf{1}_9'\otimes\mathbf{A}(x_2)\right)\mathbf{Y}\right\|^2=15.340,$$

$$S(x_1+x_2)=\left\|\left(\frac{1}{3}\mathbf{1}_9'\otimes\mathbf{A}(x_1+x_2)\right)\mathbf{Y}\right\|^2=1.823,$$

and for both groups of factors, the sum of squares would be

$$S(z_1,x_1)=\|(\mathbf{A}(z_1)\otimes\mathbf{A}(x_1))\,\mathbf{Y}\|^2=0.2039,$$

$$S(z_1,x_2)=\|(\mathbf{A}(z_1)\otimes\mathbf{A}(x_2))\,\mathbf{Y}\|^2=0.2906,$$

$$S(z_1,x_1+x_2)=\|(\mathbf{A}(z_1)\otimes\mathbf{A}(x_1+x_2))\,\mathbf{Y}\|^2=0.015,$$

$$S(z_2,x_1)=\|(\mathbf{A}(z_2)\otimes\mathbf{A}(x_1))\,\mathbf{Y}\|^2=0.3539,$$

$$S(z_2,x_2)=\|(\mathbf{A}(z_2)\otimes\mathbf{A}(x_2))\,\mathbf{Y}\|^2=0.1106,$$

$$S(z_2,x_1+x_2)=\|(\mathbf{A}(z_2)\otimes\mathbf{A}(x_1+x_2))\,\mathbf{Y}\|^2=0.7717,$$

$$S(z_1+z_2,x_1)=\|(\mathbf{A}(z_1+z_2)\otimes\mathbf{A}(x_1))\,\mathbf{Y}\|^2=0.4406,$$

$$S(z_1+z_2,x_2)=\|(\mathbf{A}(z_1+z_2)\otimes\mathbf{A}(x_2))\,\mathbf{Y}\|^2=0.3839,$$

$$S(z_1+z_2,x_1+x_2)=\|(\mathbf{A}(z_1+z_2)\otimes\mathbf{A}(x_1+x_2))\,\mathbf{Y}\|^2=0.6717,$$

$$S(z_1+2z_2,x_1)=\|(\mathbf{A}(z_1+2z_2)\otimes\mathbf{A}(x_1))\,\mathbf{Y}\|^2=1.7439,$$

$$S(z_1+2z_2,x_2)=\|(\mathbf{A}(z_1+2z_2)\otimes\mathbf{A}(x_2))\,\mathbf{Y}\|^2=1.2022,$$

$$S(z_1+2z_2,x_1+x_2)=\|(\mathbf{A}(z_1+2z_2)\otimes\mathbf{A}(x_1+x_2))\,\mathbf{Y}\|^2=8.255.$$

Since we aggregated the first [second] factors in the two groups, we had for the first and second aggregated factors the sum of squares

$$S(1)=S(z_1)+S(x_1)+S(z_1,x_1)=51.1592,$$

$$S(2) = S(z_2) + S(x_2) + S(z_2, x_2) = 35.4458.$$

The remaining former sums of squares will be grouped into the sum of squares for blocks given by

$$S(B) = S(z_1 + 2z_2) + S(x_1 + x_2) + S(z_1) + 2z_2, x_1 + x_2) = 12.7375$$

and the sum of squares for error, S. The S_1, S_2 and $S(B)$ will have five degrees of freedom each, while S will have twenty degrees of freedom.

Since S=6.2083, we have the F test statistics

$$\mathcal{F}_1 = \frac{20}{5}\frac{S_1}{S} = 32.96,$$
$$\mathcal{F}_2 = \frac{20}{5}\frac{S_2}{S} = 22.83,$$
$$\mathcal{F}_B = \frac{20}{5}\frac{S_B}{S} = 8.21$$

with p value approximately equal to zero.

We now have 20 degrees of freedom for the error instead of 25, which make it worthwhile to apply this procedure in similar situations.

Acknowledgments

This research was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) grant EP/C54171/1. Also, we are grateful to the referee of the paper for constructing comments and remarks.

References

Dey A., Mukerjee R. (1999): Fractional Factorial Plans, Wiley Series in Probability and Statistics.

Fonseca M., Mexia J.T., Zmyślony R. (2006): Binary operations on Jordan algebras and orthogonal normal models, Linear Algebra and its Applications 417: 75–86.

Jordan P., von Neumann J., Wigner E.P. (1934): On an algebraic generalization of the quantum mechanical formalism. Ann. Math. II. Ser. 35: 29–64.

Montgomery D. (2005): Design and Analysis of Experiments, 6th ed., John Wiley & Sons.

Mukerjee R., Wu C.F. (2006): A Modern Theory of Factorial Designs, Springer.

Seely J. (1971): Quadratic subspaces and completeness, the annals of Mathematical Statistics 42(2): 710–721.

Steeb W. H. (1991): Kronecker Product and Applications, Manheim.

Vanleeuwen D.M., Birkes D.S., Seely J.F. (1999): Balance and orthogonality in designs for mixed classification models. Ann. Stat., 27(6): 1927–1947